THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2017)
HW1 Solution

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1. (P.171 Q4)

   We claim that f is differentiable at 0 with $f'(0) = 0$.

   Proof of claim: Let $\epsilon > 0$ be given, choose $\delta = \epsilon > 0$. Then for all $x \in V_\delta(0) \setminus \{0\}$,

   Case 1: x is rational: then $f(x) = x^2$, and hence\epilson

$$\begin{aligned}|\frac{f(x)-f(0)}{x-0} - 0| &= |\frac{x^2}{x}| \\ &= |x| < \delta = \epsilon\end{aligned}$$

   Case 2: x is irrational: then $f(x) = 0$, and hence

$$\begin{aligned}|\frac{f(x)-f(0)}{x-0} - 0| &= 0 \\ &< \delta = \epsilon\end{aligned}$$

   Therefore, for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in V_\delta(0) \setminus \{0\}$,

$$|\frac{f(x)-f(0)}{x-0} - 0| < \epsilon$$

   Hence, f is differentiable at 0 with $f'(0) = 0$.

2. (P.171 Q10)

   For $x \neq 0$, $g(x) = x^2 \sin \frac{1}{x^2}$ is a product of functions which are differentiable at x (where $\sin \frac{1}{x^2}$ is differentiable at x by Theorem 6.16). Therefore, by Theorem 6.12, g is differentiable at x.

   For $x = 0$, we claim that g is differentiable at 0 with $g'(0) = 0$.

   Proof of claim: Let $\epsilon > 0$ be given, choose $\delta = \epsilon > 0$. Then for all $x \in V_\delta(0) \setminus \{0\}$,

$$\begin{aligned}|\frac{g(x)-g(0)}{x-0} - 0| &= |\frac{x^2 \sin \frac{1}{x^2}}{x}| \\ &= |x \sin \frac{1}{x^2}| \\ &\leq |x| < \delta = \epsilon\end{aligned}$$

Therefore, g is differentiable at 0 with $g'(0) = 0$.

Hence, g is differentiable for all $x \in \mathbb{R}$.

More explicitly, for $x \neq 0$, Chain rule gives $g'(x) = 2x \sin \frac{1}{x^2} - \frac{2\cos\frac{1}{x^2}}{x}$; for $x = 0$, $g'(0) = 0$ by above.

We also claim that g' is unbounded on $[-1, 1]$: It suffices to show that for any $M > 0$, there exists $x \in (0, 1)$ such that $|g'(x)| \geq M$.

Given any $M > 0$, choose $x \in (0, 1)$ satisfying the following inequalities:

$$\begin{cases} \frac{1}{x} > \frac{M}{2} \\ \cos\frac{1}{x^2} = 1; \quad \sin\frac{1}{x^2} = 0 \end{cases}$$

(for instance, choose $x = \frac{1}{\sqrt{2k\pi}}$, where $k \in \mathbb{N}$ is sufficiently large such that $\sqrt{2k\pi} > \frac{M}{2}$)

Then we estimate $|g'(x)|$:

$$\begin{aligned} |g'(x)| &= |2x \sin\frac{1}{x^2} - \frac{2\cos\frac{1}{x^2}}{x}| \\ &= |0 - 2\sqrt{2k\pi}| \\ &= 2\sqrt{2k\pi} \\ &> 2 \cdot \frac{M}{2} = M \end{aligned}$$

Therefore, g' is unbounded on $[-1, 1]$.

3. (P.179 Q5)

Following the hint, we consider $f(x) = x^{\frac{1}{n}} - (x-1)^{\frac{1}{n}}$ where $x \geq 1$. f is clearly continuous on $[1, +\infty)$ and differentiable on $(1, +\infty)$. Therefore, Mean Value Theorem (Theorem 6.2.4) is applicable on every finite subinterval $[1, d]$ for any $d > 1$.

For any $x > 1$, $f'(x) = \frac{1}{n}(x^{\frac{1}{n}-1} - (x-1)^{\frac{1}{n}-1})$. Since $x > x - 1 > 0$ and $\frac{1}{n} - 1 < 0$, $x^{\frac{1}{n}-1} < (x-1)^{\frac{1}{n}-1}$

Therefore, $f'(x) < 0$ for any $x > 1$.

Now given $a > b > 0$, consider $d = \frac{a}{b} > 1$. Applying Mean Value Theorem to f on $[1, d]$, there exists $c \in (1, d)$ such that

$$f(d) - f(1) = f'(c)(d-1)$$

Since $c > 1$, the above implies $f'(c) < 0$, and hence $f(d) - f(1) < 0$. Writing out the definitions explicitly, we have

$$\begin{aligned} [(\frac{a}{b})^{\frac{1}{n}} - (\frac{a}{b} - 1)^{\frac{1}{n}}] - (1 - 0) &< 0 \\ a^{\frac{1}{n}} - (a-b)^{\frac{1}{n}} &< b^{\frac{1}{n}} \end{aligned}$$

Therefore, $a^{\frac{1}{n}} - b^{\frac{1}{n}} < (a-b)^{\frac{1}{n}}$.

Remark. Many students tried to argue that $f'(x) < 0$ for $x \geq 1$, which is not true since f is actually not differentiable at $x = 1$. Even if $f'(x) < 0$ for all $x > 1$, one cannot immediately deduce that f is strictly decreasing on $(1, +\infty)$ without proving it (which is actually section 6.2 Q13). Finally, even if f is strictly decreasing on $(1, +\infty)$, it does not imply immediately that $f(1) > f(x)$ for all $x > 1$, since $1 \notin (1, +\infty)$. One has to use Mean Value Theorem to prove the final claim.